SAMPLE PAGES FROM

MODELS FOR LIFE CONTINGENCIES
A STUDY GUIDE FOR SOA EXAM MLC

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1.2. An Illustrative Example

Example 1.2.1: The following information is given:

- Methuselah (M for short) is 967 years old now. The whole number of years that M will live is a random variable K with the probability function,

  $$Pr(K = k) = (0.9)^k(0.1),\ k = 0, 1, 2, \cdots .$$

  Note that if M dies within one year, $k = 0$ and the whole number of years lived is 0.

- The annual interest rate is 25%. That means the present value of one dollar paid at time k is $(1.25)^{-k} = (0.8)^k$.

1. An insurance on M will pay 200,000 dollars if M dies during the first year, 100,000 dollars if he dies in the second or third year. The death benefit is to be paid at the end of the year of death. Furthermore, this insurer will pay an amount of 50,000 dollars at the end of 3 years if M is alive at that time. There will be no other payments. Calculate the EPV of this insurance.

   Solution: If M dies during the first year, that is, if $K = 0$, then a benefit of 200,000 is paid at time 1. So its present value is $(200,000)(0.8) = 160,000$. The probability of death in the first year is $Pr(K = 0) = 0.1$.

   If M dies during the second year, that is, if $K = 1$, then a benefit of 100,000 is paid at time 2. So its present value is $(100,000)(0.8)^2 = 64,000$. The probability of death in the second year is $Pr(K = 1) = (0.9)(0.1) = 0.09$.

   If M dies during the third year, that is, if $K = 2$, then a benefit of 100,000 is paid at time 3. So its present value is $(100,000)(0.8)^3 = 51,200$. The probability of death in the third year is $Pr(K = 2) = (0.9)^2(0.1) = 0.081$.

   If M survives three years, that is, if $K \geq 3$, then a benefit of 50,000 is paid at time 3. So its present value is $(50,000)(0.8)^3 = 25,600$. The probability that M survives three years is $Pr(K \geq 3) = 1 - 0.1 - 0.09 - 0.081 = 0.729$.

table says is that if $K = 2$, i.e., death occurs between times 2 and 3, a death benefit of 100,000 is paid at time 3 (the present value of which at time 2 is $(100,000)(0.8)$), a premium of $25,732$ is received at time 2 and expenses of 50 are incurred at time 2. Thus, if $K = 2$, then the prospective loss is $-25,732 + 50 + 100,000(0.8) = 54,318$.

If $K \geq 3$, then a benefit of $S_E = 50,000$ is paid at time 3, a premium of $\pi_2 = 25,732$ is received at time 2 and expenses of $e_2 = 50$ have been incurred at time 2. So the value of the future loss at time 2 is $50,000(0.8) + 50 - 25732 = 14,318$.

We are given that M is alive at time 2. We need then the conditional expected value of the future loss given $K \geq 2$. Given that M is alive at time 2, for $k \geq 2$, the conditional probability that $K = k$ is[3]

$$Pr(K = k|K \geq 2) = \frac{Pr(K = k)}{Pr(K \geq 2)} = \frac{(0.1)(0.9)^k}{(0.9)^2} = (0.1)(0.9)^{k-2}.$$

Therefore the conditional expected value is

$$\begin{aligned} E\left(L_2|K \geq 2\right) &= 54,318Pr(K = 2|K \geq 2) + 14,318Pr(K \geq 3|K \geq 2) \\ &= (54,318)(0.1) + (14,318)(0.9) = 18,318. \end{aligned}$$

This expected value of the future loss is called the **Reserve** or the **Policy Value**. If it is calculated using the benefit premium then it is called the **Benefit Reserve**. If the gross premium is used to calculate it, then it is called the **Gross Premium Reserve**. It is simply the expected future payments minus the expected future receipts. The insurer must have this amount in reserve to meet the future obligations.

Reserve = APV of future payments−APV of future premiums. (3)

7. Suppose that the insurer charges the gross premium as calculated in part 3, for each of n policies identical to that of M's. What is the expected amount of money that the insurer will have per expected

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[3] Note that

$$Pr(K \geq k) = (0.1)[(0.9)^k + (0.9)^{k+1} + \cdots] = (0.9)^k.$$

survivor at time 2, that is, just before the last premium is paid?

Solution: We draw a table similar to the one for the last part. Again we sit at time 2 and look at cash flow values at time 2 per policy at issue. But now we want, not the loss, but the amount of money in the fund. It should be the revenue minus the benefits and expenses.

	$K = 0$	$K = 1$	$K \geq 2$
Value of fund at time 2 (Retrospective)	$\pi_0 v^{-2}$ $-S_1 v^{-1}$ $-e_0 v^{-2}$	$\pi_0 v^{-2} + \pi_1 v^{-1}$ $-S_2$ $-e_0 v^{-2} - e_1 v$	$\pi_0 v^{-2} + \pi_1 v^{-1}$

As before, if $K = 0$ then one premium is received at time 0, an expense is incurred at time 0 and a benefit is paid at time 1. The values in the table are at time 2. With $S_1 = 200,000$, $S_2 = 100,000$, $\pi_0 = 12,866$, $e_0 = 100$, $\pi_1 = 2(12,866) = 25,732$ and $e_1 = 50$, the expected amount in the fund is

$$\begin{aligned}
& \left\{(12,866 - 100)(0.8)^{-2} - 200,000(0.8)^{-1}\right\} Pr(K = 0) \\
& + \left\{(12,866 - 100)(0.8)^{-2} + (25,732 - 50)(0.8)^{-1} - 100,000\right\} Pr(K = 1) \\
& + \left\{(12,866 - 100)(0.8)^{-2} + (25,732 - 50)(0.8)^{-1}\right\} Pr(K \geq 2) \\
= \; & (-230,055)(0.1) + (-47,951)(0.09) + 52,049(0.81) = 14,839.
\end{aligned}$$

This is per policy issued. For n policies it is $14,839n$. Now of the n individuals insured at time 0, the (expected) number of survivors is $nPr(K \geq 2) = (0.81)n$. So the amount in the fund per survivor is $14,839/0.81 = 18,320$.

It is interesting and important to note that the answers to parts 6 and 7 are extremely close. In fact, they have to be equal. (The discrepancy is due to round-off errors.) We will prove this fact later. But for now, we can understand this as follows. If an insurer starts with a sum of zero, sells insurance with the premium based on the Equivalence Principle, that means that whatever premium he expects to receive will go for whatever benefits he expects to pay plus whatever expenses he needs to pay. So the amount of money he has per survivor at any time should be just right to meet the future expected obligations.

Chapter 2

Multiple State Models

2.1. Continuous-time Markov Chains

There is a way of viewing life contingencies that is quite general and mathematically elegant. The study of life contingencies is based on the occurrence of certain events in the future. The simplest case is when there is exactly one event, such as death. An insurance pays when a person dies. An annuity pays if the person survives. Slightly more complex events are, as we saw in Examples 1.4.1 and 1.4.2, where an insurance covers two lives and pays a benefit upon the first death (or the second death) or an insurance that will have different benefits for different causes of death. In general, contingencies may involve several events such as disability, permanent or temporary, retirement and so on.

We can think of all these events in a unified manner. A person is in a certain state now (such as alive and well). At a later time he might be continue to be in be in the same state or be in another state (such as disabled or dead). A payment is made when he goes from one state to another. Let us start by saying that there are a certain number of states and we label them - rather arbitrarily - as $0, 1, 2, \cdots, n$. (You may label them any way you wish. Some authors number them as $1, 2, \cdots, n + 1$)

A payment is made when the person moves from one state to another. This movement, which we consider unpredictable or random, is called a **Transition**. In its generality - and simplicity - the one that moves need not be a person. It could be a machine, an animal, a group of people or whatever. The point is that we study a random movement that takes place from one state to another in the group of states that we have specified. We can think of this movement as a random variable's assuming one of the numbers we assigned to the states. If we denote by $Y(t)$ the random variable as a function of time, $Y(2) = 1$ means that the random variable Y assumes the value 1 at time 2. If we are studying whole life insurance on a single life and have labeled the state of being dead as 1, then $Y(2) = 1$ means the person is dead at time states, labeled, $0, 1, 2, \cdots, n$, then there are $(n+1)^2$ transition probabilities. We may represent all these probabilities together as a matrix:

$$
{}_t\mathbf{P}_x = \begin{pmatrix}
{}_tp_x^{00} & {}_tp_x^{01} & {}_tp_x^{02} & \cdots & {}_tp_x^{0j} & \cdots & {}_tp_x^{0n} \\
{}_tp_x^{10} & {}_tp_x^{11} & {}_tp_x^{12} & \cdots & {}_tp_x^{1j} & \cdots & {}_tp_x^{1n} \\
{}_tp_x^{20} & {}_tp_x^{21} & {}_tp_x^{22} & \cdots & {}_tp_x^{2j} & \cdots & {}_tp_x^{2n} \\
{}_tp_x^{i0} & {}_tp_x^{i1} & {}_tp_x^{i2} & \cdots & {}_tp_x^{ij} & \cdots & {}_tp_x^{in} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
{}_tp_x^{n0} & {}_tp_x^{n1} & {}_tp_x^{n2} & \cdots & {}_tp_x^{nj} & \cdots & {}_tp_x^{nn}
\end{pmatrix}. \tag{2}
$$

This matrix is known as the **Transition Matrix**.

It can be very helpful to think of probabilities as proportions. Think of the process being in state i as the number of persons in state i. Then ${}_tp_x^{ij}$ is the proportion of these people who are in state i now that end up in state j at time t. Then you can immediately see that

$$
{}_{t+s}p_x^{ij} = \sum_{k=0}^{n} {}_tp_x^{ik} \; {}_sp_{x+t}^{kj}. \tag{3}
$$

This can be explained as follows. Out of each person in state i aged x, ${}_tp_x^{ik}$ first go to state k over time t and then from there go to state j after a further duration s. By time t they are $x+t$ years old. Out of each one of those aged $x+t$ in state k, ${}_sp_{x+t}^{kj}$ get to state j, s units of time later. So out of ${}_tp_x^{ik}$ persons in state k, ${}_tp_x^{ik} \; {}_sp_{x+t}^{kj}$ get to state j after a duration of s. This is true for each state k. So we sum over all k. The above equation can also be written in matrix form as

$$
{}_{t+s}\mathbf{P}_x = {}_t\mathbf{P}_x \;\; {}_s\mathbf{P}_{x+t}. \tag{4}
$$

The following observation is important. Suppose a process is in state i now. Since at time t it has to be in one of the states, we have to have

$$
\sum_{j=0}^{n} {}_tp_x^{ij} = 1. \tag{5}
$$

This means the entries of each row of the transition matrix should add up to 1.

Here are a few examples.

Exercises 2.1

1. The probability that (x) will die within a year is 0.01. If he dies, the probability that the death is due to accident is 1/9-th of the probability that the death is due to non-accidental causes. Determine all the one-year transition probabilities.

2. Sally has two offices, one at home and the other downtown.

   - Given that on Monday she works at home, the probability that she works at home on Tuesday is 0.2.
   - Given that she works downtown on Monday, the probability that she works at home on Tuesday is 0.7
   - Given that she works at home on Tuesday, it is equally likely that she works at home as downtown on Wednesday
   - Given that she works downtown on Tuesday, she will definitely work at home on Wednesday.

   (a) Given that Sally works at home on Monday, calculate the probability that she works at home on Wednesday.

   (b) Given that Sally works at home on Monday, calculate the probability that she works at home Tuesday and Wednesday.

3. You are given the following information:

   - A life (x) can be in one of three states, alive and well (0), permanently disabled (2), dead (3).
   - If (x) is alive and well now, there is a 95% chance that he will be alive and well two years from now and a 4% chance that he will be disabled two years from now.
   - If (x) is disabled now, there is an 30% chance that he will be dead in two years.

   Determine all the two-year transition probabilities.

4. A life (x) can be in one of four states, alive and well (0), temporarily disabled (1), permanently disabled (2), dead (3).

Now we are ready to relate the rates of change of the transition probabilities ${}_tp_x^{ij}$ to the transition intensities. It is most convenient to use the matrix for of the transition probabilities. By definition,

$$\frac{d}{dt}\,{}_t\mathbf{P}_x = \lim_{h\to 0}\frac{{}_{t+h}\mathbf{P}_x \;-\; {}_t\mathbf{P}_x}{h}.$$

Using Eq.(5) with $s = h$, and Eqs.(8) and (9) to express ${}_hp_{x+t}^{ij}$ in terms of rates,

$$\begin{aligned}
{}_{t+h}\mathbf{P}_x &= {}_t\mathbf{P}_x\,{}_h\mathbf{P}_{x+t} \\
&= {}_t\mathbf{P}_x \begin{pmatrix} 1-h\sum_{k\neq 0}^{n}\mu_{x+t}^{0k} & h\mu_{x+t}^{01} & h\mu_{x+t}^{02} & \cdots & h\mu_{x+t}^{0n} \\ h\mu_{x+t}^{10} & 1-h\sum_{k\neq 1}^{n}\mu_{x+t}^{1k} & h\mu_{x+t}^{12} & \cdots & h\mu_{x+t}^{1n} \\ h\mu_{x+t}^{20} & h\mu_{x+t}^{21} & 1-h\sum_{k\neq 2}^{n}\mu_{x+t}^{2k} & \cdots & h\mu_{x+t}^{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ h\mu_{x+t}^{n0} & h\mu_{x+t}^{n1} & h\mu_{x+t}^{n2} & \cdots & 1-h\sum_{k\neq n}^{n}\mu_{x+t}^{nk} \end{pmatrix}.
\end{aligned}$$

We can split the right-hand side as ${}_t\mathbf{P}_x\left(\mathbf{I} + hM_{x+t}\right)$, where $\mathbf{I}$ is the identity matrix with 1's along the diagonal and 0 elsewhere and

$$\mathbf{M}_{x+t} = \begin{pmatrix} -\sum_{k\neq 0}^{n}\mu_{x+t}^{0k} & \mu_{x+t}^{01} & \mu_{x+t}^{02} & \cdots & \mu_{x+t}^{0n} \\ \mu_{x+t}^{10} & -\sum_{k\neq 1}^{n}\mu_{x+t}^{1k} & \mu_{x+t}^{12} & \cdots & \mu_{x+t}^{1n} \\ \mu_{x+t}^{20} & \mu_{x+t}^{21} & -\sum_{k\neq 2}^{n}\mu_{x+t}^{2k} & \cdots & \mu_{x+t}^{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mu_{x+t}^{n0} & \mu_{x+t}^{n1} & \mu_{x+t}^{n2} & \cdots & -\sum_{k\neq n}^{n}\mu_{x+t}^{nk} \end{pmatrix} \tag{10}$$

so that

$${}_{t+h}\mathbf{P}_x = {}_t\mathbf{P}_x + h\,{}_t\mathbf{P}_x\,\mathbf{M}_{x+t}. \tag{11}$$

Subtracting ${}_t\mathbf{P}_x$ from both sides, dividing by h and letting $h \to 0$,

$$\frac{d}{dt}\,{}_t\mathbf{P}_x = {}_t\mathbf{P}_x\,\mathbf{M}_{x+t}, \tag{12}$$

This equation is known as the **Kolmogorov Equation**.

$\mathbf{M}_{x+t}$ is the matrix whose off-diagonal entries are the intensities. Each entry along the diagonal is the negative of the sum of all rest of the intensities in that row. This makes eminent intuitive sense. Each μ_{x+t}^{ij} for $i \neq j$ is the rate at which the process goes into the state j. This rate is non-negative. The

Exercises 2.2.

1. If ${}_tp_x^{ij} = 0;\ j \neq i$, for all t, what is μ_x^{ij}?

2. Suppose there are five states. If ${}_tp_x^{50} = 0$, does it imply that $\mu_x^{50} = 0$?

3. There are four states, 0,1,2,3. Write down the Kolmogorov Equation for ${}_tp_x^{13}$ and ${}_tp_x^{\overline{11}}$

4. There are two states 0 and 1. You are given the following:

   - ${}_tp_0^{00} = 1 - 0.01t,\ 0 < t < 100$ and 0 otherwise.
   - ${}_tp_0^{10} = 0$ for all t.

   Calculate all the transition probabilities ${}_tp_x^{ij}$ and the transition intensities, μ_{x+t}^{ij}.

5. A computer can be either in use (state 0) or not in use (state 1). You are given that $\mu_x^{01} = 0.1$ and $\mu_x^{10} = 0.15$. The rates are per minute.

   (a) Given that the computer is in use now, what is the probability that it will continue to be used for next ten minutes.

   (b) Given that the computer is not use now, what is the probability that it will continue not to be used for next ten minutes.

6. For Problem 5, write the differential equations for all the probabilities of transition.

7. There are three states, 0, 1 and 2. $\mu_x^{01} = 0.01$, $\mu_x^{02} = 0.02$. All the other intensities are 0. Calculate ${}_2p_x^{00}$, ${}_2p_x^{01}$ and ${}_2p_x^{02}$.

8. There are 10 states, $0, 1, 2 \cdots, 9$. You are given that $\mu_x^{0j} = 0.001j,\ j \neq 0$. The process is in state 0 now. Calculate the probability that it will stay in the state 0 for the next 10 units of time.

9. There are three states, 0, 1 and 2. You are given that:

$$\mu_x^{01} = \begin{cases} 0.01 & 0 < x < 30 \\ 0.02 & x \geq 30 \end{cases} ;\ \mu_x^{02} = \begin{cases} 0.015 & 0 < x < 30 \\ 0.025 & x \geq 30 \end{cases}$$

   All the other intensities, $\mu_x^{ij},\ i \neq 0,\ j \neq i$ are zero.

   Calculate ${}_{35}p_{20}^{00}$ and ${}_{35}p_{20}^{01}$

Chapter 3

Two-State Model - Survival Functions for a Single Life

3.1. Continuous Model

There are two states for a life (x): 0 (alive) and 1 (dead) and there are four transition probabilities. ${}_tp_x^{00}$ is the probability that a life (x) survives t years and ${}_tp_x^{01}$ is the probability that a life (x) dies within t years. By Eq.(5) of Chapter 2 (or simply from the fact that at time t either the person is alive or is dead) ${}_tp_x^{00} + {}_tp_x^{01} = 1$.

The standard actuarial notation for these probabilities is ${}_tp_x$ and ${}_tq_x$. That is, (recall from Sec. 1.3 of Chapter 1 that T_x is the future lifetime of (x).)

$$\begin{aligned} {}_tp_x^{00} &= {}_tp_x = Pr\left(T_x > t\right) & (1) \\ {}_tp_x^{01} &= 1 - {}_tp_x = {}_tq_x = Pr\left(T_x \leq t\right) & (2) \end{aligned}$$

Since the dead cannot return to life, the transition matrix is

$$ {}_t\mathbf{P}_x = \begin{pmatrix} {}_tp_x & {}_tq_x \\ 0 & 1 \end{pmatrix}. \tag{3}$$

From Eq.(3) of Chapter 2, it is immediate that

$$ {}_{t+s}p_x = {}_tp_x \, {}_sp_{x+t}. \tag{4}$$

Another entity that is useful is the probability that (x) will survive t years and then die within s years after that. This probability is denoted by ${}_{t|s}q_x$. That is,

$$\begin{aligned} {}_{t|s}q_x &= {}_tp_x \, {}_sq_{x+t} = {}_tp_x \, {}_sq_{x+t} = {}_tp_x \left(1 - {}_sp_{x+t}\right) & (5) \\ &= {}_tp_x - {}_sp_{x+t} = Pr\left(t < T_x \leq t + s\right). & (6) \end{aligned}$$

Special symbols: The subscript 1 in the symbols ${}_1p_x$, ${}_1q_x$ and ${}_{t|1}q_x$ are omitted. Thus

$$ p_x \equiv {}_1p_x; \; q_x \equiv {}_1q_x; \; {}_{t|}q_x \equiv {}_{t|1}q_x. \tag{7}$$

The probability of survival of a newborn to time x is called the **Survival Function** and is denoted by $S(x)$. That is,

$$S(x) = {}_xp_0 = Pr(T_0 > x) = 1 - F_{T_0}(x), \tag{8}$$

where $F_{T_0}(x)$ is the cumulative distribution function of T_0, the future lifetime of a newborn. Since $F_{T_0}(0) = 0$, $F_{T_0}(\infty) = 1$ and $F_{T_0}(x)$ is a non-decreasing function, the following properties of $S(x)$ hold:

- $S(0) = 1$.
- $\lim_{x\to\infty} S(x) = 0$
- $S(x)$ is a non-increasing function.

We can use Eq.(4) to note that

$$S(x+t) = {}_{x+t}p_0 = {}_xp_0\,{}_tp_x,$$

and so

$${}_tp_x = \frac{S(x+t)}{S(x)}. \tag{9}$$

This equation lets you interpret ${}_tp_x$ as the conditional probability that a newborn will survive to age $x+t$ given that it has survived to age x. For

$${}_tp_x = \frac{S(x+t)}{S(x)} = \frac{Pr(T_0 > x+t)}{Pr(T_0 > x)}. \tag{10}$$

Note that

$${}_tp_x = Pr(T_x > t) = 1 - F_{T_x}(t).$$

By differentiating this expression we get the density function for T_x, which we denote by $f_{T_x}(t)$.

$$f_{T_x}(t) = \frac{d}{dt}F_{T_x}(t) = -\frac{d}{dt}{}_tp_x. \tag{11}$$

Rather than the transition matrix one could just as well be given the distribution of T_x.

The following relations hold.

$$\begin{aligned} S(x) &= Pr(T_0(t) > x) = \int_x^\infty f_{T_0}(s)ds & (12)\\ {}_tp_x &= Pr(T_x(t) > t) = \int_t^\infty f_{T_x}(s)ds. & (13) \end{aligned}$$

Exercises for 3.1

1. Express ${}_{t|s}q_x$ in terms of the survival function:
2. If $S(x) = e^{-\mu x}$, where μ is a positive constant, find an expression for ${}_tp_x$.
3. If T_0 is uniformly distributed over $(0, \omega)$, where ω is an integer[2],
   (a) Find $S(x)$
   (b) Determine f_{T_x}, the density function for T_x for $0 < x < \omega$.
   (c) Determine ${}_{t|s}q_x$ where $0 < x < x + t < x + t + s < \omega$.

   ω is known as the **limiting age** because the newborn will not be alive after that.
4. The density function for a newborn is

$$f_{T_0}(x) = \begin{cases} 1/100 & x \leq 25 \\ ce^{-0.02x} & x > 25. \end{cases}$$

   (a) Determine $S(x)$.
   (b) Calculate ${}_{10}p_{20}$.
   (c) Calculate ${}_{20}p_{30}$.
5. Find an expression for $f_{T_{x+t}}$, the density function for the future lifetime of $(x + t)$ in terms of f_{T_x}.
6. For the life in Problem 4, find the density function for the future lifetime of (20).
7. If $f_{T_0}(x) = c(1 + x)^{-3}$, determine ${}_tp_x$.

---

[2] This is sometimes referred to as **De Moivre's Law**.

Chapter 4

Three States - Disability Models

4.1. Permanent Disability Model - Continuous Case.

We consider a model where there are three states: 0 (alive and well), 1 (alive and disabled) and 2 (dead). Transitions from 2 to 0 and 2 to 1 are forbidden. Let us first consider the situation where state 1 corresponds to permanent disability. In that case transition from 1 to 0 is also forbidden. The transition matrix and the intensity matrix are:

$${}_t\mathbf{P}_x = \begin{pmatrix} {}_tp_x^{00} & {}_tp_x^{01} & {}_tp_x^{02} \\ 0 & {}_tp_x^{11} & {}_tp_x^{12} \\ 0 & 0 & 1 \end{pmatrix} ; \ \mathbf{M}_{x+t} = \begin{pmatrix} \mu_{x+t}^{00} & \mu_{x+t}^{01} & \mu_{x+t}^{02} \\ 0 & \mu_{x+t}^{11} & \mu_{x+t}^{12} \\ 0 & 0 & 0 \end{pmatrix}$$

Recall that $\mu_{x+t}^{ii} = -\sum_{j \neq i} \mu_{x+t}^{ij}$. The Kolmogorov equations (Eq.(12) of Chapter 2) are

$$\begin{aligned} \frac{d}{dt}\,{}_tp_x^{00} &= {}_tp_x^{00}\mu_{x+t}^{00} = -\,{}_tp_x^{00}\left(\mu_{x+t}^{01} + \mu_{x+t}^{02}\right) && (1) \\ \frac{d}{dt}\,{}_tp_x^{01} &= {}_tp_x^{00}\,\mu_{x+t}^{01} + {}_tp_x^{01}\,\mu_{x+t}^{11} = {}_tp_x^{00}\,\mu_{x+t}^{01} - {}_tp_x^{01}\,\mu_{x+t}^{12} && (2) \\ \frac{d}{dt}\,{}_tp_x^{02} &= {}_tp_x^{00}\,\mu_{x+t}^{02} + {}_tp_x^{01}\,\mu_{x+t}^{12} && (3) \\ \frac{d}{dt}\,{}_tp_x^{11} &= {}_tp_x^{11}\,\mu_{x+t}^{11} = -\,{}_tp_x^{11}\,\mu_{x+t}^{12} && (4) \\ \frac{d}{dt}\,{}_tp_x^{12} &= {}_tp_x^{11}\,\mu_{x+t}^{12} && (5) \end{aligned}$$

The last two equations are precisely those we saw in the last chapter for two states,alive and dead. The solutions are

$$\begin{aligned} {}_tp_x^{11} &= \exp\left\{-\int_0^t \mu_{x+s}^{12}\,ds,\right\} && (6) \\ {}_tp_x^{12} &= 1 - \exp\left\{-\int_0^t \mu_{x+s}^{12}\,ds\right\}. && (7) \end{aligned}$$

Chapter 6

Several States, Multiple Decrements

6.1. Continuous Case

We now consider a situation where a failure occurs and that failure could be due to several different causes or decrements. For instance, the failure could be death and the decrements could be accident or some cause other than accident. Or the failure could be termination of employment and the decrements could be resignation, dismissal, retirement or death.

In all these cases there are $n+1$ states, $0, 1, 2, \cdots, n$. If the process is in any state other than 0, it cannot go to any other state. If 0 is being alive then any other state is failure due to a decrement and so there is no escape from any state other than 0. The intensity matrix is (recall that the sum of the entries in each row has to be zero)

$$\mathbf{M}_x = \begin{pmatrix} -\sum_{j\neq 0}^{n} \mu_x^{0j} & \mu_x^{01} & \cdots & \mu_x^{0j} & \cdots & \mu_x^{0n} \\ \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since $\mu_x^{ij} = 0.$ for $i \neq 0$, we might as well drop the superscript 0 and write

$$\mu_x^{0j} = \mu_x^{(j)}.$$

The Kolmogorov equations for this model are

$$\frac{d}{dt}\,{}_tp_x^{00} = -\,{}_tp_x^{00} \sum_{j=1}^{n} \mu_{x+t}^{(j)} \tag{1}$$

$$\frac{d}{dt}\,{}_tp_x^{0j} = {}_tp_x^{00}\mu_{x+t}^{j},\ j = 1, 2, \cdots, n. \tag{2}$$

need not equal 1. However,

$$\begin{aligned}\sum_{j=1}^{n}\int_0^{\infty} f_{T,J}(t,j)dt &= \int_0^{\infty} {}_tp_x^{(\tau)}\sum_{j=1}^{\infty}\mu_{x+t}^{(j)}dt\\ &= \int_0^{\infty}\left(-\frac{d}{dt}\right){}_tp_x^{(\tau)}dt = 1.\end{aligned}$$

The marginal density function for the time of failure is

$$f_T(t) = \sum_{j=1}^{n} f_{T,J}(t,j) = {}_tp_x^{(\tau)}\sum_{j=1}^{n}\mu_{x+t}^{(j)} = {}_tp_x^{(\tau)}\,\mu_{x+t}^{(\tau)} \tag{16}$$

From Eqs.(15) and (16), we can derive the following result. Given that failure occurs at time t the conditional probability that the cause of failure is j is

$$\begin{aligned}Pr(J=j|T=t) &= f_{J|T}(j|t) = \frac{f_{T,J}(t,j)}{f_T(t)}\\ &= \frac{\mu_x^{(j)}(t)\;{}_tp_x^{(\tau)}}{\mu_x^{(\tau)}(t)\;{}_tp_x^{(\tau)}} = \frac{\mu_x^{(j)}(t)}{\mu_x^{(\tau)}(t)}\end{aligned} \tag{17}$$

Example 6.1.1: For a double decrement model, you are given:

$$\begin{aligned}\text{(i)}\ \ \mu_{10}^{(1)}(t) &= \frac{1}{30-t},\ 0\le t<30\\ \text{(ii)}\ \ \mu_{10}^{(\tau)}(t) &= \frac{1}{30-t}+\frac{1}{20-t},\ 0\le t<20\end{aligned}$$

Calculate the probability that (10) will terminate from cause 2 during the 6-th year.

Solution:

$$\begin{aligned}{}_tp_x^{(\tau)} &= \exp-\int_0^t \mu_{10}^{(\tau)}(s)\,ds\\ &= \exp\{\ln(30-t)+\ln(20-t)-\ln 30-\ln 20\}\\ &= \left(\frac{30-t}{30}\right)\left(\frac{20-t}{20}\right)\\ f_{T,J}(t,2) &= \mu_{10}^{(2)}(t)\;{}_tp_x^{(\tau)} = \frac{30-t}{600},\ 0<t<20\\ Pr(5<t<6, j=2) &= \int_5^6 \frac{30-t}{600}\,dt\\ &= \frac{25^2-24^2}{1200} = 0.04083.\end{aligned}$$

10.2. A Recursion Relation

An extremely useful recursion relation between reserves in successive years can be derived from a very simple idea. It is also extremely general. It is nothing more than accounting. (In fact, most of what we will be doing in the rest of the book will be accounting.) The basic idea is that if you have a certain amount of money at time k what you would have at time $k+1$ is just that amount plus interest minus what you would spend during the year.

Let us be a bit more formal now. Suppose for an an insurance on (x) the loss variable at time k is L_k. It does not matter what sort of insurance it is except that the benefit is payable at the end of the year of death. The premium may or may not include expenses. The premium may be determined on any basis whatever. The premiums need not be level. Even the interest rate need not be the same from year to year. In fact we will denote by i_k the interest rate and $v_k = (1+i_k)^{-1}$ the discount factor for the year $(k, k+1)$. The only things we assume are:

- A premium of P_k is paid at time k.
- The premium related expenses at time k are e_k.
- If the insured dies during the year $(k, k+1)$, an amount of S_{k+1} will be paid at time $k+1$.
- The expenses related to the benefit payment at time $k+1$ is E_{k+1}.
- There is no cash flow in the interval $(k, k+1)$.

The loss variable at time k, namely L_k, should equal the value at time k of L_{k+1} plus that of the benefit paid at time $k+1$ plus that of the expenses associated with the benefit payment plus the premium related expenses at time k minus the premium paid at time k.

Now there are two possibilities; either the insured dies within the year $(k, k+1)$ (the probability of which is q_{x+k}) or he survives (with probability p_{x+k}). In the first case, $L_{k+1} = 0$, because there are no future premiums or no future liabilities. If the insured survives, there is no benefit paid nor are there expenses associated with the benefit payment. Thus

$$L_k = \begin{cases} v(S_{k+1} + E_{k+1}) + e_k - P_k & \text{with probability } q_{x+k} \\ v\,{}_{k+1}L + e_k - P_k & \text{with probability } p_{x+k}\ . \end{cases}$$

Continuous case, Thiele's Differential Equation: It is an easy matter now to go from Eq.(9) to the continuous case. All we have to do is change the yearly period to a period of infinitesimally small duration. That is, replace k by a continuous variable t and $k+1$ by $t+h$. Thus we will make the following replacements:

- The reserve at time k is replaced by the value at time t., i.e., ${}_kV$ by ${}_tV$.
- The reserve, ${}_{k+1}V$, a duration of a year later is replaced by ${}_{t+h}V$.
- P_k, and e_k are premiums and premium related expenses for one year. Thus they are yearly rates. So for a duration of h we replace them by $P_t h$ and $e_t h$.
- S_{k+1} and E_{k+1} are the one-time benefit and the one time expense associated with the benefit, paid at time $k+1$. So we replace them with S_{t+h} and E_{t+h}.
- Finally, we replace $1 + i_k$, which is the accumulation over one year by $e^{\delta_t h}$, the accumulation over a duration of h.[1]

Now from Eq.(8) of Chapter 2, since ${}_tp_x^{01} = {}_tq_x$, ${}_hq_{x+t} = \mu_{x+t}h$ and ${}_hp_{x+t} = 1 - \mu_{x+t}\,h$. Finally we write for infinitesimally small h,

$$e^{\delta_t h} = 1 + \delta_t\, h, \;\; S_{t+h} = S_t + S_t' h, \;\; \text{and} \;\; E_{t+h} = E_t + E_{t+h}'\, h.$$

Putting all this in Eq.(10) and neglecting terms quadratic in h, since h is infinitesimally small, we get

$$\begin{aligned}
&({}_tV + P_t h - e_t h)\,(1 + \delta_t\, h) \\
&= {}_tV + \delta_t\, {}_tV\, h + (P_t - e_t)\, h \\
&= (S_t + S_t' h + E_t + E_t' h)\, \mu_{x+t}\, h + {}_{t+h}V\,(1 - \mu_{x+t}\, h) \\
&= (S_t + E_t)\, \mu_{x+t}\, h + {}_{t+h}V - {}_tV\, \mu_{x+t}\, h.
\end{aligned}$$

In the last line we have set ${}_{t+h}Vh$ to ${}_tVh$ since for infinitesimally small h, ${}_{t+h}V - {}_tV$ is proportional to h and so ${}_{t+h}V\,h - {}_tV\,h$ is proportional to h^2, which we neglect.

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[1] Strictly speaking this should be $\exp\{\int_0^h \delta_s ds\} = e^{\delta_t h}+$ terms much smaller than h.

Chapter 11. Asset Shares, Profit Testing, Universal Life Insurance

11.1. Asset Shares

The reason that I have lumped these topics together is that they constitute little more than bookkeeping and in that they are similar. The main theme is how to account for the funds the insurer has, compare it with what he would need for the future and estimate profits. The calculations are just slightly more complex forms of the recursive calculation of reserves.

At the end of Section 10.1 we introduced the idea of retrospective policy value. It was defined as the expected amount of money that the insurer has per survivor. If the premium is calculated using the Equivalence Principle and the retrospective and prospective policy values are calculated on the same basis as the premium, then the two policy values will have to be equal. Otherwise the two need not.

The **Asset Share** is the actual amount of money that the insurer has at a certain time per policy. It is based on the mortality experience and actual interest up to that time. IF the mortality experience and other factors such as the interest rate are the same as those on the basis of the reserve calculation, then the asset share and the retrospective policy value will be equal. The symbol for the asset share at time k is AS_k.

Example 11.1.1: A ten year deferred annuity-due is issued to a person aged 50. Starting 10 years from now it will pay 10,000 at the beginning of every year that the annuitant is alive. A single gross premium will be paid at the time of issue. If the annuitant dies during the deferral period the single premium will be returned without interest at the end of the year of death. You are given the following information:

- Mortality follows the ILT.
- $i = 0.06$.
- The expenses are